COMMUNICATIONS AND COMMENTS

Accuracy of Approximations for CIELAB Chroma and Hue Difference Computation

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Received 11 November 1995; accepted 21 March 1996

Abstract: The transformation in CIELAB from differences in the L, a*, b* coordinates to those in lightness, chroma, and hue, ΔL^*, ΔC^*_{ab}, ΔH^*_{ab}, can be approximated by a rotation in 3-space. Expressions for the error in the approximation of chroma and hue differences are developed. Significant errors are introduced if either the hue angle or chroma difference between reference and sample colors are large. A computed example illustrates the use of the analysis.* © 1997 John Wiley & Sons, Inc. *Col Res Appl, 22, 61–64, 1997.*

Key words: CIELAB; color difference; chroma difference; hue difference; color-measurement accuracy

INTRODUCTION

Sluban and Nobbs[1] approximate the transformation of CIELAB color differences from the standard, ΔL^*, Δa^*, Δb^* to lightness, chroma, and hue differences, ΔL^*, ΔC^*_{ab}, ΔH^*_{ab}, as a rotation about the L^* axis. This is given by

$$\begin{bmatrix} \Delta L^* \\ \Delta C^*_{ab} \\ \Delta H^*_{ab} \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos h_{ab} & \sin h_{ab} \\ 0 & -\sin h_{ab} & \cos h_{ab} \end{bmatrix} \begin{bmatrix} \Delta L^* \\ \Delta a^* \\ \Delta b^* \end{bmatrix}, \quad (1)$$

where h_{ab} is the hue angle of the reference color. Equation (1) rotates the differences[2] in Δa^* and Δb^* by the angle $-h_{ab}$. The rotated differences are taken as equal to ΔC^*_{ab} and ΔH^*_{ab}, respectively.

This simple matrix form lends itself readily to the analysis of error propagation between the two CIELAB representations.[3] From multivariate statistics given a set of signals, ΔL^*, Δa^*, Δb^*, with error described by the covariance matrix, $\Sigma_{\Delta L^*\Delta a^*\Delta b^*}$, the equivalent covariance for the lightness, chroma, and hue difference values is

$$\Sigma_{\Delta L^*\Delta C^*\Delta H^*} = \mathbf{M}\Sigma_{\Delta L^*\Delta a^*\Delta b^*}\mathbf{M}^T,$$

where ^{T} represents matrix transpose,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \mu_{h_{ab}} & \sin \mu_{h_{ab}} \\ 0 & -\sin \mu_{h_{ab}} & \cos \mu_{h_{ab}} \end{bmatrix},$$

and $\mu_{h_{ab}}$ is the mean hue angle

$$\mu_{h_{ab}} \approx \tan^{-1}\left(\frac{\mu_{b^*}}{\mu_{a^*}}\right).$$

In this report we investigate the accuracy of the rotation Eq. (1), but first review the equations for CIELAB chroma and hue differences:

$$\Delta C^*_{ab} \quad \text{and} \quad \Delta H^*_{ab}.$$

Given two sets of CIELAB color coordinates $\mathbf{P}(L^*_1,$

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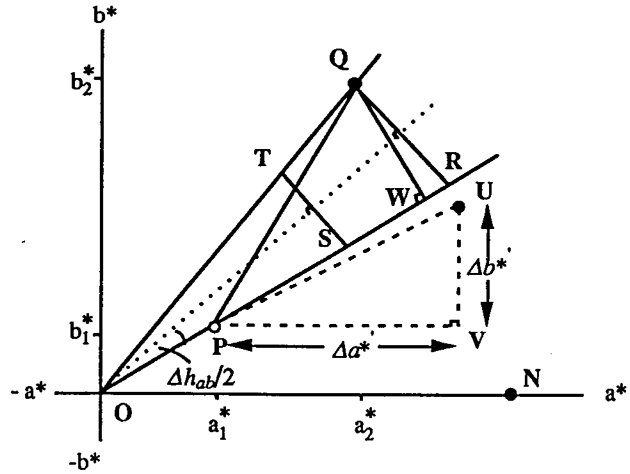

FIG. 1. Interpretation of Eq. (1) as a rotation of the vector (Δa^*, Δb^*) about point **P**, with the rotated differences, $\Delta a^{*\prime}$ (**PV**) and $\Delta b^{*\prime}$ (**UV**) as approximations to ΔC^*_{ab} and ΔH^*_{ab}. Point **P** is the reference, **Q** is the sample, and the dotted line bisects ∠**QOP**.

a_1^*, b_1^*) and Q(L_2^*, a_2^*, b_2^*), the chroma value of each is[4]

$$C_1^* = \sqrt{a_1^{*2} + b_1^{*2}};\ C_2^* = \sqrt{a_2^{*2} + b_2^{*2}}, \qquad (2)$$

and the chroma difference is

$$\Delta C^*_{ab} = C_2^* - C_1^*. \qquad (3)$$

The hue difference is given by

$$\Delta H^*_{ab} = \sqrt{\Delta E^{*2}_{ab} - \Delta L^{*2} - \Delta C^{*2}_{ab}}, \qquad (4)$$

or[5,6]

$$\Delta H^*_{ab} = 2\sqrt{C_1^* C_2^*}\,\sin\left(\frac{\Delta h_{ab}}{2}\right), \qquad (5)$$

where Δh_{ab} is the hue-angle difference

$$\Delta h_{ab} = \tan^{-1}\left(\frac{b_2^*}{a_2^*}\right) - \tan^{-1}\left(\frac{b_1^*}{a_1^*}\right). \qquad (6)$$

The geometrical interpretation of Eqs. (2)–(6) is shown in Fig. 1, where $C_1^* =$ **OP**, $C_2^* =$ **OQ**, $\Delta H^*_{ab} =$ **TS**, and $\Delta h_{ab} =$ ∠**NOQ** − ∠**NOP**.

The above equations define the chroma and hue differences, and it is with these that we now compare the matrix calculation of Eq. (1). Before doing so, it is worth noting that no small-angle assumption is necessary for Eq. (5) to hold. If that assumption is made, however, then

$$\Delta H^*_{ab} \approx \Delta h_{ab}\sqrt{C_1^* C_2^*},$$

if Δh_{ab} is expressed in radians.

MATRIX ROTATION

As stated above, the approximation of Eq. (1) represents a rotation of the (Δa^*, Δb^*) coordinates over an angle $-h_{ab}$. The Δa^* and Δb^* distances in the rotated coordinates, $\Delta a^{*\prime}$ and $\Delta b^{*\prime}$, are then taken as the values for ΔC^*_{ab} and ΔH^*_{ab}, i.e.,

$$\begin{bmatrix} \Delta L^* \\ \Delta C^*_{ab} \\ \Delta H^*_{ab} \end{bmatrix} \approx \begin{bmatrix} \Delta L^* \\ \Delta a^{*\prime} \\ \Delta b^{*\prime} \end{bmatrix} = \mathbf{M}\begin{bmatrix} \Delta L^* \\ \Delta a^* \\ \Delta b^* \end{bmatrix},$$

as shown in Fig. 1. The true value of ΔC^*_{ab} is

$$\Delta C^*_{ab} = \mathbf{OQ} - \mathbf{OP}.$$

The rotated difference, $\Delta a^{*\prime}$(**PV**), is seen as related to chroma difference, following the derivation of expressions for the approximation errors given in the Appendix. From Eq. (A7),

$$\Delta a^{*\prime} = \Delta C^*_{ab} - 2C_2^*\sin^2\left(\frac{\Delta h_{ab}}{2}\right),$$

where the second term of the RHS is the approximation error. The error is a bias given by

$$\epsilon_C = a^{*\prime} - \Delta C^*_{ab} = -2C_2^*\sin^2\left(\frac{\Delta h_{ab}}{2}\right), \qquad (7)$$

where the negative sign indicates that $\Delta a^{*\prime}$ underestimates ΔC^*_{ab}. The relationship between ϵ_C and Δh_{ab} is indicated in Fig. 2.

In Fig. 1, we also see that $\Delta b^{*\prime}$ (**VU**) is not equal to ΔH^*_{ab} (**ST**). From the Appendix, Eq. (A4),

$$\Delta b^{*\prime} = \frac{\Delta H^*_{ab} C_2^*}{\sqrt{C_1^* C_2^*}}\cos\left(\frac{\Delta h_{ab}}{2}\right).$$

We can express the hue-angle approximation error as a factor,

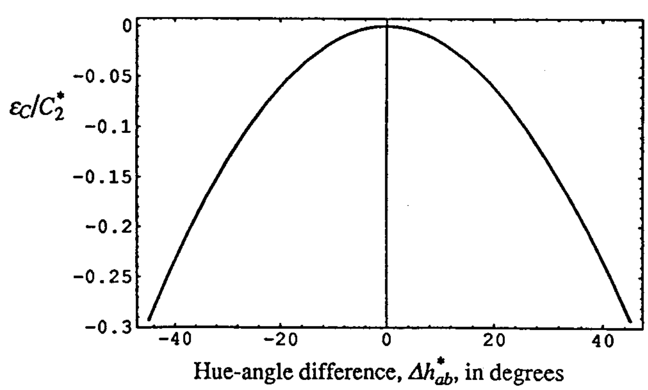

FIG. 2. The chroma-difference bias error term plotted vs. the difference in hue angle.

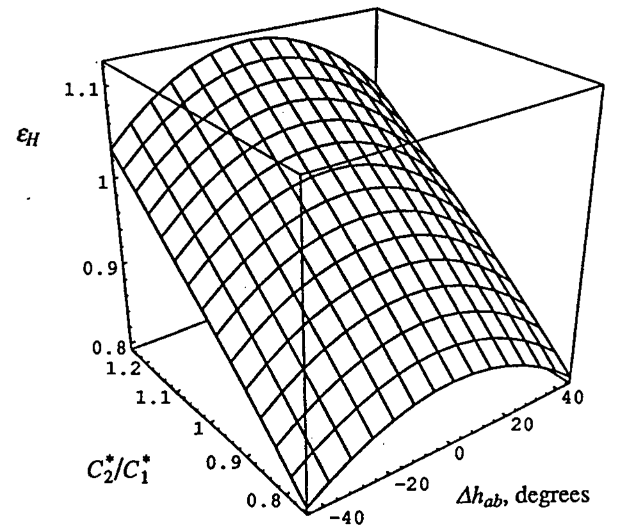

FIG. 3. The hue-difference error factor, ε_H, plotted vs. Δh_{ab} and C_2^*/C_1^*.

$$\epsilon_H = \frac{\Delta b^{*\prime}}{\Delta H^*_{ab}} = \frac{C_2^*}{\sqrt{C_1^* C_2^*}} \cos\left(\frac{\Delta h_{ab}}{2}\right), \qquad (8)$$

which is shown plotted in normalized form in Fig. 3. Equations (7) and (8) provide corrections for the rotated difference measures so that they are equal to ΔC^*_{ab} and ΔH^*_{ab},

$$\Delta C^*_{ab} = \Delta a^{*\prime} - \epsilon_{C^*_{ab}}, \quad \Delta H^*_{ab} = \frac{\Delta b^{*\prime}}{\epsilon_H}. \qquad (9)$$

COMPUTED EXAMPLE

Consider the color difference defined by the reference color and sample with CIELAB a^*, b^* coordinates (1, 1) and (2, 3), respectively, as shown in Fig. 1. The corresponding chroma, hue angle, chroma difference, and hue difference, as computed by Eqs. (2)–(6), are given in Table I.

If we compute the approximation of Eq. (1) with the rotation angle, $h_{ab} = 45°$, then

$$\Delta a^{*\prime} = 2.121, \ \Delta b^{*\prime} = 0.707.$$

Applying the corrections due to the rotation approximation, Eq. (9),

$$\Delta C^*_{ab} = \Delta a^{*\prime} + 2C_2^* \sin^2\left(\frac{\Delta h_{ab}}{2}\right) = 2.191,$$

and

$$\Delta H^*_{ab} = \frac{\Delta b^{*\prime}\sqrt{C_1^* C_2^*}}{C_2^* \cos\left(\frac{\Delta h_{ab}}{2}\right)} = 0.445,$$

consistent with the values listed in Table I.

For this example color difference we see that the rotation approximation introduces a large error into the hue-difference estimate. From Eq. (8), this is because of the large relative difference between C_1^* and C_2^*, rather than the hue-angle difference, 11.3°. In cases where the hue-angle difference is large, the approximations for both chroma and hue difference will contain large errors.

CONCLUSIONS

The description of the transformation between ΔL^*, Δa^*, Δb^*, and ΔL^*, ΔC^*_{ab}, ΔH^*_{ab} using the matrix Eq. (1) can be described as a rotation in 3-space. This description is useful, due to the constant-coefficient nature of the matrix, in such areas as formulation sensitivity analysis and error propagation. The matrix rotation, however, is an approximation to the true transformation in CIELAB. By considering the geometrical nature of the transformation, the magnitude of the approximation error has been presented in Eqs. (7) and (8). The form of these equations indicates that the key contributors to the errors are the hue-angle difference between the reference and sample, and the magnitude of the sample chroma.

The usefulness of the approximation for the propagation of noise or error statistics can also be understood by examining Eqs. (7) and (8). For distributions around chromatic mean colors, the hue-angle difference can often be assumed to be small, and, therefore, the error is small. As shown in the computed example, the hue difference is also dependent on the factor, $C_2^*/\sqrt{C_1^* C_2^*}$. If we are examining small differences between the color and members of an associated distribution, this term will be small, except when C_2^* approaches zero.

APPENDIX: ERROR EXPRESSIONS FOR THE APPROXIMATIONS FOR ΔC^*_{ab} AND ΔH^*_{ab}

Referring to Fig. 1, the approximation of Eq. (1) represents the rotation of the vector **PQ** about the point **P** over the angle $-h_{ab}$, i.e., **Q** is rotated to **U**. The distances **PV** and **VU** are taken as approximations for ΔC^*_{ab} and ΔH^*_{ab}, respectively.

First note that **PV** is parallel to the a^* axis. Since **QP**

TABLE I. Chroma and hue data for the color difference example.

	Reference	Sample color
(a^*, b^*)	(1, 1)	(2, 3)
chroma C_1^*, C_2^*	1.414	3.606
hue angle $h_{1_{ab}}$, $h_{2_{ab}}$	45.0°	56.3°
	Value	**Approximation**
ΔC^*_{ab}	2.191	2.121
ΔH^*_{ab}	0.445	0.707

was rotated by $-h_{ab}$ around point **P**, $\angle$**QPS** = $\angle$**UPV**, **PQ** = **PU**, and it follows that Δ**QPW** $\cong$ Δ**UPV** (ASA). Therefore,

$$\mathbf{PV} = \mathbf{PW} \quad \text{and} \quad \mathbf{VU} = \mathbf{WQ}, \qquad (A1)$$

(CPCTC). It can also be shown that $\angle$**WQR** = $\Delta h_{ab}/2$ and

$$\mathbf{RQ} = \frac{\mathbf{WQ}}{\cos\left(\frac{\Delta h_{ab}}{2}\right)}. \qquad (A2)$$

Δ**TOS** and Δ**QOR** are similar (AAA), so it follows that

$$\mathbf{ST} = \mathbf{RQ}\,\frac{\mathbf{TO}}{\mathbf{QO}}. \qquad (A3)$$

Substituting Eqs. (A1) and (A2) into (A3),

$$\mathbf{ST} = \frac{\mathbf{VU}}{\cos\left(\frac{\Delta h_{ab}}{2}\right)}\,\frac{\mathbf{TO}}{\mathbf{QO}},$$

and, from Fig. 1,

$\mathbf{ST} = \Delta H^*_{ab}$, $\mathbf{VU} = \Delta b^{*\prime}$,

$$\mathbf{TO} = \sqrt{C^*_1 C^*_2}, \quad \text{and} \quad \mathbf{QO} = C^*_2, \qquad (A4)$$

$$\Delta H^*_{ab} = \frac{\Delta b^{*\prime}}{\cos\left(\frac{\Delta h_{ab}}{2}\right)}\,\frac{\sqrt{C^*_1 C^*_2}}{C^*_2}.$$

ΔC^*_{ab} is given by distance **PR**. As in Eq. (A1),

$$\mathbf{WR} = \mathbf{QR}\sin\left(\frac{\Delta h_{ab}}{2}\right). \qquad (A5)$$

QR is normal to the bisector of the angle Δh_{ab}, so

$$\mathbf{QR} = 2\mathbf{OQ}\sin\left(\frac{\Delta h_{ab}}{2}\right). \qquad (A6)$$

If we substitute Eq. (A6) into (A5),

$$\mathbf{PR} = \mathbf{PW} + 2\mathbf{OQ}\sin^2\left(\frac{\Delta h_{ab}}{2}\right),$$

or, since $\mathbf{PW} = \Delta a^{*\prime}$,

$$\Delta C^*_{ab} = \Delta a^{*\prime} + 2C^*_2 \sin^2\left(\frac{\Delta h_{ab}}{2}\right). \qquad (A7)$$

ACKNOWLEDGMENTS

I thank Roy Berns for bringing the matrix approximation to my attention, Kevin Burns for his comments on geometry notation, and Lisa Reniff for reviewing the manuscript.

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