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Wiener Spectrum Estimation at Zero Frequency via Direct Digital Computation

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Abstract Previous methods for Wiener spectrum estimation have not accounted for the unknown average density of most imaging systems. As a result, estimates of the zero frequency value are unavailable or are measured separately. Here we describe a method that uses short data sequences to derive estimates of the entire Wiener spectrum, including the zero frequency value. The estimates approach the true Wiener spectrum as the number of data sequences is increased. Computer simulation results are shown to be consistent with the analysis. In addition, the effect of image density nonuniformity (due to, for example, development variation) on the Wiener spectrum estimates is addressed. Two statistical tests for the detection of density nonuniformity are described.

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Introduction

The Wiener spectrum has for some time been used to describe image noise, and several reviews of the literature are available.[1,2] Since the Wiener spectrum is the Fourier decomposition of optical density fluctuations, it is a statistical description of image noise. The task of measuring the Wiener spectrum is one of statistical estimation, what is generally called spectral or spectrum estimation.

Spectrum estimation has been studied in the context of zero mean, or known mean random processes.[3] However, imaging systems generally involve a nonzero and unknown mean value. The application of the usual Wiener spectrum estimation methods in this case results in no estimate at zero spatial frequency. An equivalent problem occurs when estimating the autocovariance function.[4–6] The very low frequency (zero frequency) Wiener spectrum value can be calculated from separate measurements made with a large sampling aperture. The term "zero frequency" estimate here refers to an estimate representing the Wiener spectrum value in a spatial frequency range close to zero. The Wiener spectrum value at zero frequency represents the integral of the autocovariance function, by transform properties.

Our purpose here is to investigate estimates of the zero frequency Wiener spectrum that can be calculated when the entire one-dimensional spectrum is measured. The procedure is a modification of a direct digital method using short data sequences.[7] These estimates are shown, under certain conditions, to be equivalent to the large aperture granularity measurement.

To evaluate the utility of the estimates, the form of the bias error is obtained by consideration of the expected values of the estimates. A computer simulation experiment shows results consistent with the bias analysis and also the form of the variance of the estimates. The analysis and simulation results would not strictly apply to images that are not uniformly exposed and developed. To detect density nonuniformity, two statistical tests are described that can be applied when using the above Wiener spectrum measurement procedure.

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Wiener Spectrum Estimation

For isotropic noise processes, the two-dimensional autocovariance function (acf) and Wiener spectrum are sufficiently represented as functions of single variables. The one-dimensional acf is defined as[2]

$$\gamma(l) = L\Delta X E\{[D(n) - \mu][D(n + l) - \mu]\}$$

where $E[\cdot]$ is the statistical expectation, and μ is the mean of $D(n)$, image density fluctuations. The measurement slit length is L and the sampling distance is ΔX. The autocorrelation function is

$$R(l) = L\Delta X\, E[D(n)D(n + l)]$$

and

$$\gamma(l) = R(l) - \mu^2$$

The Wiener spectrum will be defined as the Fourier transform of the acf,

$$W(j) = \lim_{N\to\infty} L\Delta X \sum_{k=-N}^{N} \gamma(k)e^{-i2\pi jk/N} \tag{1}$$

for a discretely sampled process, where $i = \sqrt{-1}$. The index j corresponds to the spatial frequency index $j/N\Delta X$. Each value of the discrete function W represents the Wiener spectrum within the bandwith determined by the spatial frequency sampling. The zero frequency Wiener spectrum value, therefore, describes the spectrum over the interval $(0,1/N\Delta X)$.

The noise process whose Wiener spectrum is to be estimated is assumed to be ergodic[7] and to have zero covariance beyond a distance $N\Delta X$. The first and second statistical moments are, therefore, constant and can be written

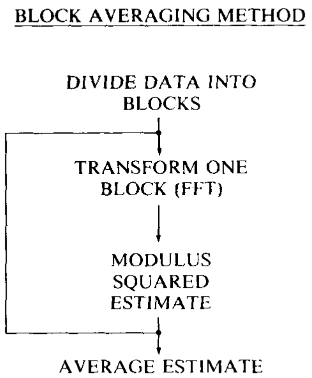

Figure 1. The block averaging method for Wiener spectrum estimation.

$$\left.\begin{aligned} E[D(k)] &= \mu \\ \mathrm{Cov}[D(n), D(n+k)] &= \gamma(k) \end{aligned}\right\} \quad \text{for all } k \qquad (2)$$
$$\gamma(k) = 0 \qquad \text{for all } k \geq N,$$

where Cov[·] is the covariance function.

To estimate the Wiener spectrum,[9] the density data trace, $D(n)$, is divided into several short sequences (blocks) and the discrete Fourier transform of each block is computed as shown in Fig. 1. The square of the modulus of the Fourier transform is the estimator of the Wiener spectrum. The Fourier transform is calculated for each data block and the squared moduli are averaged at each spatial frequency to reduce the variance of the estimate. The calculation is given by

$$\widehat{W}_1(j) = \frac{L\Delta X}{NM} \sum_{m=1}^{M} \left| \sum_{n=1}^{N} D(n,m) e^{-i2\pi jn/N} \right|^2 \qquad (3)$$

where M is the number of blocks and $|\cdot|$ indicates the modulus of a complex number.

Most imaging systems have nonzero mean density levels; hence, some estimate of the mean must be subtracted from the data. If the sample mean is subtracted from each data block before the Fourier transform is computed, the Wiener spectrum estimate at zero frequency is necessarily zero. This is because the sum of the deviations from the sample mean is identically zero, and there is no zero frequency estimate available in this case. From this it might be inferred that it is not possible to obtain a zero frequency Wiener spectrum estimate via direct calculation. However, if the estimate of the mean that is subtracted from each data point is the sample mean for the entire data trace, the above procedure does give a zero frequency Wiener spectrum estimate, which we will show.

If no account is taken of the mean value of the density fluctuations, the estimate $\hat{W}_1(j)$, which is a random variable, has an expected value (see the appendix),

$$E[\widehat{W}_1(j)] = L\Delta X \left[\sum_{k=-(n-1)}^{N-1} \gamma(k) \times \left(1 - \frac{|k|}{N}\right) e^{-i2\pi jk/N} + \delta(j) N\mu^2 \right] \qquad (4)$$

where $\delta(j)$ is the Dirac delta function. The term $(1 - |k|/N)$ is due to the finite data block length and is referred to as the lag window. The use of various other windows[10] will not be addressed; however, the estimation procedure is valid for all common windows (but the above term will differ for each). As the block length, N, increases, the first term of the RHS (right-hand side) of Eq. (4) approaches that of Eq. (1), if one notes the properties of Eqs. (2). The second term of the RHS of Eq. (4) is the positive bias due to the nonzero mean value, μ, and only effects the estimate for $i = 0$.

The estimate $\hat{W}_1(0)$ could be corrected by subtracting $L\Delta x N\mu^2$, but one rarely has *a priori* knowledge of μ. In its place, the average of all data points in the entire trace, $\overline{D}$, can be substituted. This gives

$$\widehat{W}_2(0) = \frac{L\Delta X}{N} \left[\frac{1}{M} \sum_{m=1}^{M} \left(\sum_{n=1}^{N} D(n,m) \right)^2 \right] - L\Delta X N \overline{D}^2$$
$$= \widehat{W}_1(0) - L\Delta X N \overline{D}^2 \qquad (5)$$

where

$$\overline{D} = \frac{1}{NM} \sum_{k=1}^{MN} D(k) \qquad (6)$$

The expected value of $\widehat{W}_2(0)$ can be shown to be (see the Appendix)

$$E[W_2(0)] = L\Delta X \left[\sum_{k=-(N-1)}^{N-1} \gamma(k) \left(1 - \frac{|k|}{N}\right) - \frac{1}{M} \sum_{k=-(N-1)}^{N-1} \gamma(k) \left(1 - \frac{|k|}{MN}\right) \right] \qquad (7)$$

The first term of the RHS of Eq. (7) is identical to that of Eq. (4) for a zero mean random process. The second term of Eq. (7) is a bias that can be made arbitrarily small by increasing the number of data blocks. Thus $\widehat{W}_2(0)$ would yield a value that approaches the true zero frequency Wiener spectrum. A practical drawback of this estimate is that subtraction of a term of large magnitude may lead to computer round-off errors unless double precision arithmetic is used.

An alternative correction for the average density, more amenable to single precision arithmetic, is one that corrects each data point by subtracting $\overline{D}$ from all the data points, before computing the discrete Fourier transform. This estimator, $\widehat{W}_3(j)$, is given by

$$\widehat{W}_3(j) = \frac{L\Delta X}{NM} \sum_{m=1}^{M} \left| \sum_{n=1}^{N} (D(n,m) - \overline{D}) e^{-i2\pi jn/N} \right|^2 \qquad (8)$$

The expected value of $\widehat{W}_3(0)$ is identical to that of $\widehat{W}_2(0)$ given in Eq. (7), as shown in the Appendix.

As an example of this Wiener spectrum measurement method, Fig. 2 shows an estimate, $\widehat{WS}_3(u)$, based on actual measured data. The aperture used was 25 μm × 1.0 mm with a sampling distance of 25 μm. The trace length was 2000 data points and the calculation was based on 40 blocks of 50 density readings per block.

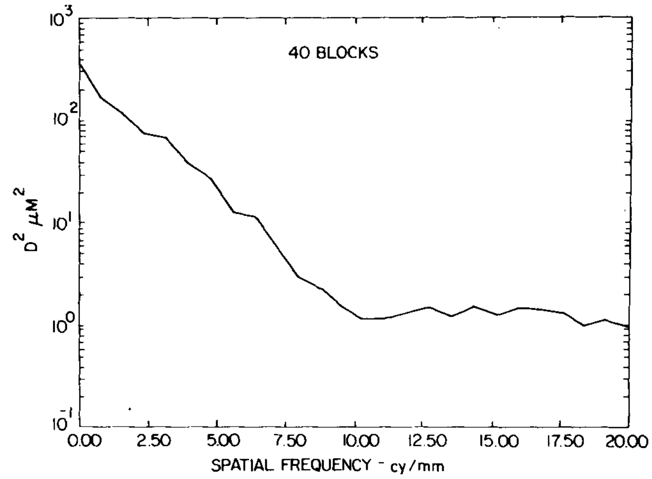

Figure 2. Example of Wiener spectrum estimate, $\widehat{WS}_3(u)$, based on actual measured density data.

Approximate Bias of Zero Frequency Estimates

The bias of the Wiener spectrum arises from two sources as shown by Eq. (7). One is the finite data block length and the other is the presence of the unknown mean, $\overline{D}$. The term $(1 - |k|/N)$ of Eqs. (4) and (7) corresponds to the convolution of the Wiener spectrum with a spectral window.[10] This introduces a bias into the spectrum estimate at all frequencies, except for a white noise (constant spectrum) process. This bias can be reduced by increasing the number of data points in each block, N, but remains for all estimates based on a finite data record.

The estimate at zero frequency contains an additional bias given in Eq. (7) by

$$-\frac{L\Delta X}{M}\sum_{k=-(N-1)}^{N-1}\gamma(k)\left(1-\frac{|k|}{MN}\right)$$

To determine the approximate magnitude of this "zero frequency" bias, some simplifying assumptions are made.[11] We assume that $\gamma(k) = 0$ for k much less than N, which says that the data block length is large compared to the correlation length. Under this assumption Eq. (7) becomes

$$E[\widehat{W}_{2,3}(0)] \approx L\Delta X\left[\sum_{k=-(N-1)}^{N-1}\gamma(k)\left(1-\frac{|k|}{N}\right)\right]\left[1-\frac{1}{M}\right] \quad (9)$$

The zero frequency Wiener spectrum estimate is negatively biased (i.e., less than its true value) by a factor of approximately $1/M$. This suggests an approximate bias correction given by

$$\widehat{W}^*_{2,3}(0) = \frac{M}{M-1}W_{2,3}(0) \quad (10)$$

In practice, however, this zero frequency bias is small because M is usually large (50 or more), in order to reduce the variance of the estimates of the entire Wiener spectrum.

Comparison with $A\sigma_D^2$

The low frequency Wiener spectrum can also be estimated from independent data recorded with a large sampling aperture. The noise parameter estimate

$$\widehat{W}(0) = A\sigma_D^2 = \frac{A}{M}\sum_{m=1}^{M}(D_m-\overline{D})^2 \quad (11)$$

approaches the true Wiener spectrum, $W(0)$, for a large aperture and a large value of M.[12]

To investigate the relationship between $\widehat{W}_{2,3}(0)$ and $\widehat{W}(0)$, consider Eq. (8), which can be written

$$\widehat{W}_3(0) = L\Delta XN\left[\frac{1}{M}\sum_{m=1}^{M}\left(\frac{1}{N}\sum_{n=1}^{N}(D(n,m)-\overline{D}\right)^2\right] \quad (12)$$

The summation over n and the division by N computes the average of the optical density readings within each data block. If the fluctuations are small compared to the mean density, this is approximately equivalent to single, large aperture density measurements. Under this assumption, Eq. (12) is seen as computing the sample variance from M data points. The effective aperture, corresponding to the scanning and aperture, A, is $L\Delta XN$ and, therefore,

$$\widehat{W}_3(0) \cong \widehat{W}(0)$$

for small fluctuations and where the correlation distance is much less than $N\Delta X$. Thus the large aperture measurement is identically equivalent to the zero frequency estimate, $W_3(0)$, with a sample size equal to the number of data blocks.

Computed Example

To illustrate the use of the Wiener spectrum estimation method, $\widehat{W}_3$ was calculated in a computer simulation experiment as shown in Fig. 3. Using a random number generator, a Gaussian random process (sequence) was generated with a mean density value of 0.5. This simulated density trace was then Fourier transformed via the F.F.T. and multiplied by a

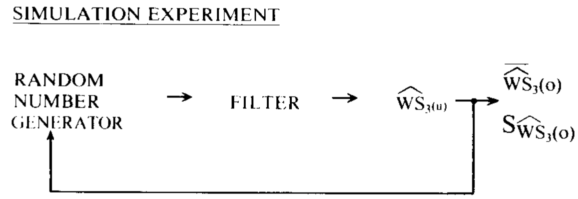

Figure 3. The computer simulation experiment to compute the Wiener spectrum estimate, $\widehat{WS}_3(u)$.

transfer function so as to have a known Wiener spectrum,

$$W(u) = \frac{1}{\left(1+\frac{u}{B}\right)^2} \quad (13)$$

where B is a constant. The filtered data were then inverse transformed to give the simulated random density trace whose Wiener spectrum was to be estimated.

To conform to typical electrophotographic image measurement conditions, a sampling distance, ΔX, of 25 μm and a sampling aperture of 25 μm × 1.0 mm were assumed. For this sampling distance, the Wiener spectrum can be estimated over the frequency range 0–20 mm^{-1}.[13] A block length of $N = 50$ density data points was used, giving spectrum estimates at increments of 0.8 mm^{-1}. The Wiener spectrum estimate, $\widehat{W}_3$, was calculated using various numbers of data blocks, M. For each value of $M, \widehat{W}_3$ was calculated twenty times from different simulated density traces. The bias and standard deviation of the Wiener spectrum estimates at each frequency were then calculated from the twenty values.

The results for the low frequency estimate are given in Table I. As the number of data blocks per estimate is increased, the mean value of $\widehat{W}_3(0)$ approaches the true spectrum value indicating the reduction in bias predicted by Eq. (7). Using the bias correction approximation of Eq. (10), a closer agreement with the true Wiener spectrum was obtained as shown in the third column.

Although the standard deviation of the zero frequency estimates has not been explicitly derived, it was estimated in the simulation experiment. Column 4 of Table I gives the calculated coefficient of variation (c.v.), which is the standard deviation divided by the estimate mean. We compare this with the error in the large aperture estimate $\widehat{W}(0)$. The c.v. of $\widehat{W}(0)$ is given by

$$\text{c.v.} = \sqrt{\frac{2}{M-1}}$$

under the assumption of a normal distribution where the random variable $(M-1)\,s^2/\sigma^2$ is distributed as χ^2 with $M-1$ degrees of freedom.[14] Comparing columns 4 and 5 shows close agreement and supports the analogy between $\widehat{W}_3(0)$ and $A\sigma_D^2$.

Detection of Image Nonuniformity

The Wiener spectrum estimates and analysis given assume uniform density images whose microdensitometric fluctuations are statistically stationary.[8] This may not be true if exposure or development vary over the sample image. However, depending on the degree of nonuniformity, a useful estimate can still be obtained.

Wiener spectrum estimation in the presence of nonuniformities (nonstationarity) has been approached in several ways, including the use of special-purpose hardware,[15] trend removal,[5,16] and filtering of the data.[17] Curve fitting can successfully remove certain types of nonuniformities prior to calculation of the estimate. Routine spatial filtering of the data, however, often biases the estimate—whether or not the image is nonuniform.

An alternative approach is to first detect the presence of image nonuniformity before applying any corrective tech-

TABLE I. Results of Computer Simulation Experiment[a]

Number of blocks per estimate, M	$\hat{W}_3(0)$ mean, μm^2D^2	Bias corrected mean — $\hat{W}_3^*(0)$, μm^2D^2	Estimated standard deviation, or coefficient of variation, μm^2D^2	$\sqrt{\frac{2}{M-1}}$
10	0.90	1.00	0.428	0.417
20	0.90	0.95	0.334	0.324
40	0.98	1.01	0.279	0.226
60	0.99	1.01	0.173	0.184
80	0.98	1.00	0.149	0.159

[a] Mean and standard deviation are based on 20 Wiener spectrum estimates, $\hat{W}_3(0)$. The data sequence (block) length, N = 50 points, true W(0) = 1.0 μm^2D^2 and $\bar{D}$ = 0.5.

niques. To this end, two nonparametric statistical tests are now described that detect a changing mean or variance over the image. After the image is sampled, the mean and variance of the optical density data in each block are estimated via the usual calculations. Both a sign and run test[18] are then applied to the sets of block sample means and variances to detect image nonuniformity.

Sign Test

This test can be applied to the sets of block sample means or block variances. Given the average of the set of data (block means or variances values) stationarity implies that half of the values would be above the mean of the set and half below. For a set of M blocks, assuming a binomial distribution, one would expect the number of set values above the mean to be M/2 with a variance of $M^2/4$. If a is the actual number of block values above the mean, the statistic

$$Z = \frac{(a - M/2) - 1/2}{(M^2/4)^{1/2}}$$

is approximately normally distributed for large M.

Run Test

Here the quantity to be counted is the number of runs (groups) of data above and below the mean. From the theory of probability, the expected number of runs from a sample size M is

$$\mu = \frac{2ab}{M} + 1$$

with a variance of

$$\sigma^2 = \frac{2ab(2ab - M)}{M^2(M - 1)}$$

where a and b are the number of values (mean or variance) above and below the mean value, respectively, $(a + b = M)$. If R is the total number of runs observed, then

$$Z = \frac{R - \mu - 1/2}{\sigma}$$

is approximately normally distributed.

For both tests, any critical value and associated confidence level can be chosen from a table of standard normal probabilities. For the run test used with small samples (few data blocks), binomial probabilities and exact tables are available.[19] Since the tests are applied to the sets of block mean and block variance estimates, a test failure gives evidence that the mean or variance of the density readings is changing over the image.

To understand the effect of image nonuniformity on the Wiener spectrum estimates, consider an image whose local average density varies across the sample. The nonuniformity can be thought of as a very low frequency signal (trend) added to the particle-formed image fluctuations. The calculated Wiener spectrum estimate would be positively biased by the square of the Fourier transform modulus of the trend signal. In addition, a greater (error) variance may be associated with $\hat{W}S(0)$ than for a measurement of a uniform sample. Nonuniformity of the mean density alone has little effect on Wiener spectrum estimates at spatial frequencies greater than zero, unless the nonuniformity has a period on the order of a data block length or less. This follows from the sampling theorem.

Nonuniformity of the measured variance is more difficult to visualize, but its detection implies changing image density fluctuations over the sample. This can often occur in conjunction with a nonuniform mean density, since for most imaging systems, signal and noise levels are related.[20] The variance of a density trace is equal to the integral of the Wiener spectrum over all spatial frequencies. The Wiener spectrum estimate for a sample of nonuniform variance, therefore, has a greater error at several or all frequencies than for a measurement of a uniform noise sample.

Conclusion

A method has been presented for estimating the zero frequency Wiener spectrum of an image of unknown average density. The modulus squared estimate, calculated over short data sequences, can be modified to estimate this low frequency spectrum value. One can remove the sample mean from the data before calculating the estimate, $\hat{W}_3(0)$, or correct the zero frequency value afterwards, $\hat{W}_2(0)$. These zero frequency estimates are biased; however, for practical measurements the bias is small, given by 1 − 1/number of blocks. Under certain conditions, the zero frequency estimates approach the large aperture noise measurement $A\sigma_D^2$.

The simulation experiment results were consistent with the analysis of the bias and also indicated the magnitude of the standard deviation of $\hat{W}_3(0)$. The estimated standard deviation was found to be approximately a fraction (2/number of blocks)$^{1/2}$ times the true Wiener spectrum value. Increased errors would be expected, however, for measurements of nonuniform images. Detection of image density nonuniformity is possible via the application of the nonparametric run and sign tests to the sets of block mean and variance values. These tests can be applied to determine whether data should be corrected (for example by trend removal) for systematic effects, such as exposure or development variations.

Appendix

Expected Value of $\hat{W}_1(0)$, $\hat{W}_2(0)$, $\hat{W}_3(0)$

$\hat{W}_1(0)$

Equation (3) gives the form of $\widehat{W}_1(0)$ where no correction is made for a nonzero mean value

$$\widehat{W}_1(0) = \frac{L\Delta X}{NM}\left[\sum_{m=1}^{M}\left(\sum_{n=1}^{N} D(n,m)\right)^2\right] \qquad (1a)$$

Again we make the same assumption regarding the noise process that are given in Eq. (2), then

$$E[\widehat{W}_1(0)] = \frac{L\Delta X}{N} E\left[\left(\sum_{n-1}^{N} D(n)\right)^2\right]$$
$$= L\Delta X\left[Var\left(\sum_{n-1}^{N} Dn\right) + \left[E\left(\sum_{n=1}^{N} D(n)\right)\right]^2\right] \qquad (2a)$$

Using Eq. (2), we find that

$$E\left[\sum_{n=1}^{N} D(n)\right] = N\mu \tag{3a}$$

and

$$\mathrm{Var}\left(\sum_{n=1}^{N} D(n)\right) = \sum_{n=1}^{N} \mathrm{Var\,D(n)}$$

$$+ 2 \sum_{n=1}^{N-1} \sum_{l=1}^{N-1} \mathrm{Cov}\,(D(n),D(l))$$
$$(n>l)$$

$$= N \sum_{k=-(N-1)}^{N-1} \gamma(k) \left(1 - \frac{|k|}{N}\right). \tag{4a}$$

This result can be visualized in a matrix notation by summing all elements of the autocovariance matrix for a block of data.[21] Using results (3a) and (4a), the expected value of $\widehat{W}_1(0)$ is

$$E[\widehat{W}_1(0)] = L\Delta X \left[\sum_{k=-(N-1)}^{N-1} \gamma(k) \left(1 - \frac{|k|}{N}\right) + N\mu^2\right]. \tag{5a}$$

$\hat{W}_2(0)$

The estimator $\widehat{W}_2(0)$ is defined in Eqs. (5) and (6). To find the expected value, we apply the results of Eqs. (2), (3a) and (4a)

$$E[\overline{D}] = \mu$$

$$\mathrm{Var}\,[\overline{D}] = \frac{1}{M^2N^2}\, Var \left[\sum_{m=1}^{M} \sum_{n=1}^{N} D(n,m)\right] \tag{6a}$$

$$= \frac{1}{MN} \sum_{k=-(MN-1)}^{MN-1} \gamma(k) \left(1 - \frac{|k|}{MN}\right)$$

$$= \frac{1}{MN} \sum_{k=-(N-1)}^{N-1} \gamma(k) \left(1 - \frac{|k|}{MN}\right) \tag{7a}$$

The results (6a) and (7a) are used to derive the expected value of $\widehat{W}_2(0)$:

$$E[\widehat{W}_2(0)] = E[(W_1(0)] - L\Delta X N E[\overline{D}^2]$$

$$= L\Delta X \left[\sum_{k=-(N-1)}^{N-1} \gamma(k) \left(1 - \frac{|k|}{N}\right) + N\mu^2\right.$$

$$\left. - N[E[\overline{D}]^2\ \mathrm{Var}\,[\overline{D}]]\right.$$

$$= L\Delta X \left[\sum_{k=-(N-1)}^{N-1} \gamma(k) \left(1 - \frac{|k|}{N}\right)\right.$$

$$\left. - \frac{1}{M} \sum_{k=-(N-1)}^{N-1} \gamma(k) \left(1 - \frac{|k|}{MN}\right)\right]. \tag{8a}$$

$\hat{W}_3(0)$

An alternative correction for the process mean is given by Eq. (8). The expected value of this estimator can be expanded into the form,

$$E[\widehat{W}_3(0)] = \frac{L\Delta X}{MN} \sum_{m=1}^{M} \left[\mathrm{Var}\left[\sum_{n=1}^{N} D(n,m)\right] + N^2\, \mathrm{Var}\,[\overline{D}]\right.$$

$$\left. - 2N\, \mathrm{Cov}\left[\sum_{n=1}^{N} D(n),\overline{D}\right]\right. \tag{9a}$$

The variance terms of the RHS of Eq. (9a) are given by Eqs. (4a) and (7a). The covariance summation is expanded as described by Fuller[22] for each of the M data blocks and after (much) manipulation of the equivalent covariance matrix, it can be reduced to

$$\mathrm{Cov}\left[\sum_{n=1}^{N} D(n),\overline{D}\right] = \sum_{k=-(N-1)}^{N-1} \gamma(k) \left(1 - \frac{|k|}{MN}\right) \tag{10a}$$

Substituting the results of Eqs. (4a), (7a), and (10a) into Eq. (9a) yields the same result as Eq. (8a), i.e.,

$$E[\widehat{W}_2(0)] = E[\widehat{W}_3(0)]$$

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