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# VERIFICATION OF A METHOD TO ESTIMATE THE WIENER KERNEL TRANSFORMS OF A NONLINEAR SYSTEM

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ABSTRACT

A method for estimating the Fourier transforms of the Wiener kernels of a nonlinear system is presented. In order to verify the method, it is applied to various systems in several computer simulation experiments. The estimates for the Fourier transforms of the first, second and third kernels are obtained and found to converge to the theoretical kernel transforms. The number of calculations necessary, and the estimate variance, is considered in comparison with a time domain method for Wiener kernel estimation.

## 1.0 INTRODUCTION

In general, a system is specified by an input — output relationship. Knowledge of the output which is due to a known input is available if one has a mathematical description of the system. Such a description is a useful tool for both analysis and design.

The linear time invariant system has been extensively studied and is usually described by its impulse response or, in the frequency domain, the transfer function. The system description used here is a generalization of the impulse response, the set of Wiener kernels. Only single valued, shift (time) invariant, single input-single output systems will be addressed.

Since nonlinear systems occur frequently in many practical applications, it is appropriate to discuss examples of where the Wiener kernel description may be of value. Consider a transistor whose dc transfer characteristics are parabolic rather than linear. Instead of concerning oneself with nonlinear system analysis one may choose to merely specify the percent total harmonic distortion. Harmonic distortion will, however, vary with input signal amplitude and therefore does not completely describe the system response. A similar situation occurs in the large area (low frequency) characteristics of many imaging systems [1]. Instead of input and output being voltage and current values (as with the previous case), one deals with optical reflectance, transmittance and density values.

Nonlinear input-output characteristics are also found in the study of biological systems, where the Wiener system description has been successfully applied [2,3].

The output of a nonlinear system of the above type which is due to a white Gaussian input can be described by the Wiener series expansion [4]. Wiener represents the system output by the orthogonal expansion,

$$y(t) = \sum_{n=0}^{\infty} G_n\,[k_n, x(t)] \tag{1}$$

where $\{k_n\}$ is the set of Wiener kernels, $\{G_n\}$ is the set of orthogonal functions and $x(t)$ is the input. The first four terms (functionals) of the expansion are:

$$G_o[k_o, x(t)] = k_o$$

$$G_1[k_1, x(t)] = \int k_1(\tau)x(t-\tau)d\tau$$

$$G_2[k_2, x(t)] = \iint k_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2$$

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$$- C\int k_2(\tau_1,\tau_1)d\tau_1 \qquad (2)$$

$$G_3[k_3,x(t)] = \iiint k_3(\tau_1,\tau_2,\tau_3)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)d\tau_1 d\tau_2 d\tau_3$$

$$- 3C\iint k_3(\tau_1,\tau_2,\tau_2)x(t-\tau_1)d\tau_1 d\tau_2$$

where the power density spectrum of the input is C watts/Hz. The limits of integration are to be taken from $-\infty$ to $+\infty$ unless otherwise indicated. Each term of the series is seen as an n-dimensional convolution integral operating on the input and a generalized impulse response, the Wiener kernel of order n. Wiener shows that the Wiener kernels are, or can be made symmetrical with respect to their arguments, so

$$\left.\begin{aligned} k_2(\tau_1,\tau_2) &= k_2(\tau_2,\tau_1) \\ k_3(\tau_1,\tau_2,\tau_3) &= k(\tau_1,\tau_3,\tau_2) \end{aligned}\right\} \qquad (3)$$

The same property holds for the Fourier transforms of the Wiener kernels.

The objective of this work is to verify a method for estimating the Fourier transforms of the Wiener kernels of a nonlinear system. The form of the estimates was first noticed by French and Butz [5]. The method is applied to various systems in several computer simulation experiments and the results shown. The variance of the kernel transform estimate and the computation required are also considered. A brief discussion of a time domain kernel estimation method precedes the kernel transform estimates.

## 2.0 WIENER KERNEL AND KERNEL TRANSFORM ESTIMATES

### 2.1 Wiener Kernel Transform Estimates

Lee and Schetzen [6] present a method for estimating the Wiener kernels by cross-correlation. They introduce a set of functionals formed from delay circuits with a white Gaussian input. The first four Wiener kernel estimates are

$$\left.\begin{aligned} k_o &= E[y(t)] \\ k_1(\tau) &= \frac{1}{C}E[y(t)x(t-\tau)] \\ k_2(\tau_1,\tau_2) &= \frac{1}{2C^2}E[y(t)x(t-\tau_1)x(t-\tau_2)] \\ &\text{for } \tau_1 \neq \tau_2 \\ k_3(\tau_1,\tau_2,\tau_3) &= \frac{1}{6C^3}E[y(t)x(t-\tau_1)x(t-\tau_2)x(t-\tau_3)] \\ &\text{for } \tau_1 \neq \tau_2, \tau_2 \neq \tau_3, \tau_1 \neq \tau_3 \quad . \end{aligned}\right. \qquad (4)$$

where E indicates the expected value. Successful application of this method has been reported [2,3].

### 2.2 Wiener Kernel Transform Estimates

The Fourier transform of the Wiener kernel of order n is given by

$$K_n(w_1,\ldots,w_n) = \int\cdots\int k_n(\tau_1,\ldots,\tau_n)e^{-j(w_1\tau_1+\ldots+w_n\tau_n)}\,d\tau_1\ldots d\tau_n \ . \qquad (5)$$

The estimates for the first-, second- and third-order kernel transforms are [ 7],

$$\left.\begin{aligned} K_1(w) &= \frac{E[X^*(w)Y(w)]}{C} \\ K_2(w_1,w_2) &= \frac{E[X^*(w_1)X^*(w_2)Y(w_1+w_2)]}{2C^2} \\ &\text{for } w_1 \neq -w_2 \end{aligned}\right. \qquad (6)$$

$$K_3(w_1, w_2, w_3) = \frac{E[X^*(w_1)X^*(w_2)X^*(w_3)Y(w_1+w_2+w_3)]}{6C^3} \quad (6)$$

$$\text{for } w_1 \neq -w_2, w_2 \neq -w_3, w_1 \neq -w_3.$$

where * indicates the complex conjugate. For the second- and third-order estimates, impulse functions occur when any two arguments sum to zero.

## 3.0 SIMULATION EXPERIMENTS TO ESTIMATE THE WIENER KERNEL TRANSFORMS

The first-, second-, and third-order Wiener kernel transforms were estimated for several systems. The diagram in Fig. 1 outlines the experimental procedure which involved simulating a nonlinear system response to a discrete, white, Gaussian input.

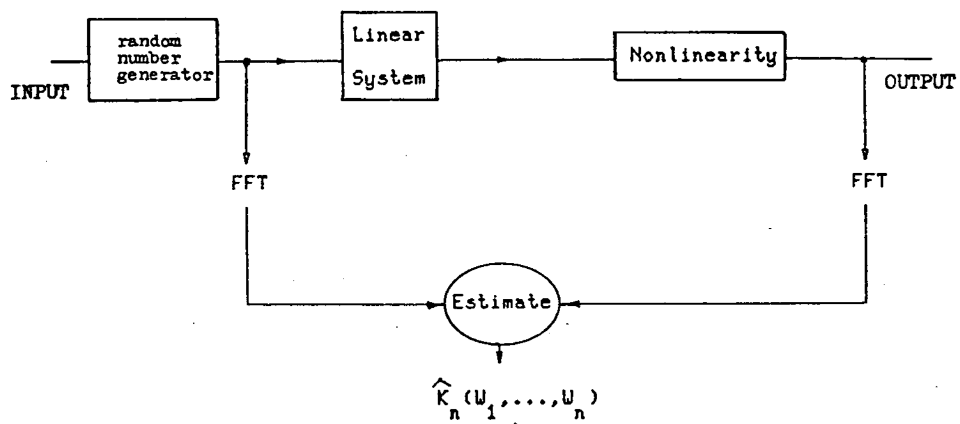


FIGURE 1. Outline for kernel transform estimation experiments.

The experiments were repeated many times since the expected values of the estimates should approach the kernel transforms. The estimates necessitate the discrete Fourier transform of both input and output signals (sequences) and this is calculated via the fast Fourier transform algorithm [8].

The first-order kernel transform was estimated for two recursive (I.I.R) digital filters of first (low-pass) and second (resonator) order. The first kernel transform is merely the linear transfer function. The magnitude of the calculated and estimated kernel transform after 1000 averages for the resonator is given in Fig. 2. The frequency is normalized so w=1 is the Nyquist frequency.

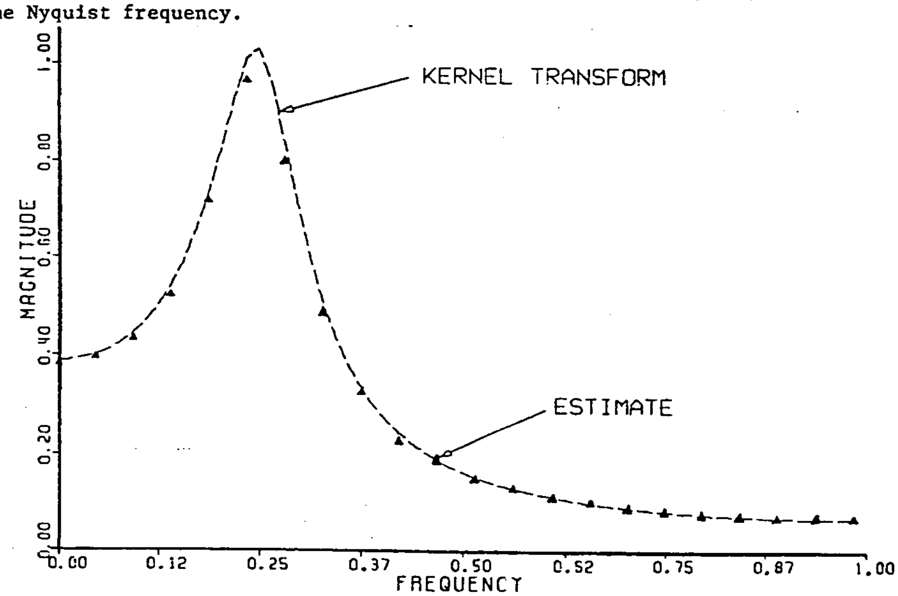


FIGURE 2. First-order estimate magnitude after 1000 averages-resonator.

Experiments to verify the second Wiener kernel transform estimate were performed. The nonlinear systems were of two types; the first consisted of the pervious filters whose output was squared as shown in Fig. 3.

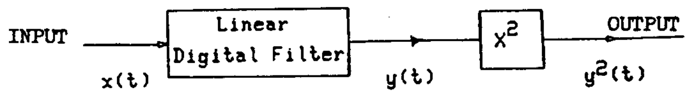


FIGURE 3. System configuration used in second-order kernel transform experiments.

The second-order kernel transform for a system of the above form is [9]

$$K_2(w_1,w_2) = T(w_1)T(w_2) \tag{7}$$

where T(w) is the transfer function of the appropriate digital filter. A perspective plot of the magnitude of the kernel transform estimate after 2500 averages, for the resonator nonlinear system, is given in Fig. 4. The second type was a three-stage system consisting of a linear subsystem cascaded with the output of the system of Fig. 3. The second-order estimates approached $K(w_1,w_2)$ except where $w_1 = -w_2$, as expected from equation (6).

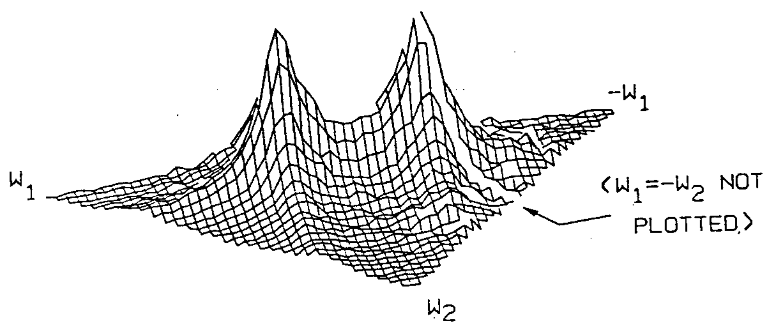


FIGURE 4. Second-order estimate magnitude after 2500 averages — resonator nonlinear system.

Two experiments were performed to verify the third-order kernel transform estimate. The two nonlinear systems consisted of the linear filters whose output was cubed. Again, the estimates approached the calculated kernel transforms.

## 4.0 VARIANCE OF THE ESTIMATE AND COMPUTATION NECESSARY

During the experiments of Section 3.0, an estimate of the variance of the kernel transform estimate was obtained. The estimate of the estimate magnitude variance was

$$\hat{\sigma}^2(w_1,..,w_n) = \frac{\sum_{i=1}^{N} \left| \hat{K}_n^i(w,..,w_n) - \bar{K}_n(w_1,..,w_n) \right|^2}{N-1} \tag{8}$$

where $\hat{K}_n^i$ is the ith estimate of order n and $\bar{K}_n$ is the estimate after N (complex) averages.

The expression (8) was calculated for several points of estimates. For the first kernel, $\hat{\sigma}$ approached the kernel magnitude value. The value of $\hat{\sigma}$ approached approximately four and twelve times the kernel magnitude value for the second and third kernels. This suggests the estimate standard deviation is one, four and twelve times the magnitude for n=1,2,3.

## 4.1 COMPUTATION NECESSARY FOR WIENER KERNEL ESTIMATION

The original sampling of x(t) and y(t) specifies the Nyquist frequency. This restricts the frequencies over which one can estimate $K_n$ via equation (6) since one has no information of $Y(w_1+...+w_n)$ for values of the argument greater than the Nyquist frequency. This can be shown to reduce the necessary computation by a factor of n![5]. The expression for the approximate number of real multiplications needed for the kernel transform estimate of order n, real input/output of length N, noting symmetry [7], is

$$2\left[N \log_2 \frac{N}{2} + 2N + \frac{n N^n}{(n!)^2}\right] . \tag{9}$$

If the variance of the single estimates are inferred from the experimental results, the number of averages needed for a given average estimate variance is known. This follows if the variance is inversely proportional to the number of independent estimates averaged, as for the power spectrum [10]. This is given in Table 1 for 5% variance, as is the approximate number of real multiplications for N=64.

TABLE 1

COMPARISON OF MULTIPLICATION NECESSARY FOR VARIOUS ESTIMATES (DATA LENGTH 64, 5% VARIANCE)

| Estimate | No. of Averages | No. of Real Mult. | No. of Points of Estimate | Mult/Point | Ratio |
|---|---|---|---|---|---|
| Power Spectrum | 20 | $1.0 \times 10^4$ | 64 | 160 | 1 |
| 1st Kernel Transform | 20 | $1.8 \times 10^4$ | 64 | 280 | 1.8 |
| 2nd Kernel Transform | 320 | $16 \times 10^5$ | 2048 | 780 | 4.8 |
| 3rd Kernel Transform | 2880 | $12.8 \times 10^7$ | $4.4 \times 10^4$ | $2.8 \times 10^3$ | 17.5 |

A final comparison is made of the relative efficiency of the time domain and frequency domain estimates. Consider the number of real multiplications needed for the discrete estimate of the nth kernel of the form of equation (4). The estimate requires approximately [7]

$$\frac{MN^n}{(n-1)!} \qquad (10)$$

real multiplications, exploiting kernel symmetry, where M is the entire record and N is the (shift) length of the estimate.

Lee and Schetzen [6] estimated the second Wiener kernel of a nonlinear system. The kernel estimate was from data, M, equal to 30,000. That is, each point of the estimate was the result of 30,000 averages and achieved an RMS error of 0.6% of the maximum value. A similar experiment was performed with the second kernel transform estimate. The mean squared error was found inversely proportional to the number of averages. After 100 averages the MS error was found to be 3.55%. Extrapolating, it would take 98,600 averages to reach an RMS error of 0.6%.

Consider the number of real multiplications required for the second kernel estimate for M equal to 30,000 and the second-order kernel transform estimate averaged 98,600 times for N equal to 128. From equation (10) the number of real multiplications required for the cross-correlation estimate is $4.92 \times 10^8$. The number of multiplications needed for each kernel transform estimate from equation (9) is $1.84 \times 10^4$. This must be calculated 98,600 times which requires $1.81 \times 10^9$. The approximate ratio of real multiplications for time/frequency domain methods is 1/3.6, for this example.

## 5.0 CONCLUSIONS

To verify the Wiener kernel transform estimate method, it has been applied to several nonlinear systems in a series of computer simulation experiments. The systems consisted of feed-through interconnections of linear and nonlinear subsystems. The estimates of the Fourier transforms of the first, second and third kernels were obtained and found to approach the theoretical kernel transforms.

Experimental results suggest that the single kernel transform estimate standard deviation is one, four and twelve times the value of the kernel transform for the first, second and third kernels, respectively.

As a measure of the computation, the approximate number of real multiplications necessary for the kernel transform estimates has been considered. For a data length of 64 and 5% variance the first, second, and third kernel transform estimates need 1.8, 4.8 and 17.5 times as many real multiplications per estimate point as does the power spectrum estimate. The number of real multiplications needed substantially increased with increased order n and data length. For comparison of the time and frequency domain second Wiener kernel estimates the number of real multiplications necessary for each to achieve a desired RMS error was calculated. The large variance associated with the kernel transform estimate required approximately 3.6 times more multiplications.

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